Calogero-Sutherland Particles as Quasisemions

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ABSTRACT

The ultraviolet structure of the Calogero-Sutherland models is examined, and, in particular, semions result to have special properties. An analogy with ultraviolet structures known in anyon quantum mechanics is drawn, and it is used to suggest possible physical consequences of the observed semionic properties.

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Recently, there has been renewed interest[1-8] in the Calogero-Sutherland models[9-11], especially in connection with the study of fractional *exclusion* statistics[4,7,12-16] in 1+1 dimensions. These quantum mechanical models describe particles whose dynamics is governed by a Hamiltonian of the form

$$-\sum_i \frac{d^2}{dx_i^2} + \sum_{i<j} \frac{\pi^2 \beta(\beta-1)}{L^2 sin^2\left(\pi(x_i-x_j)/L\right)} + V(\{x_i\}) \ , \tag{1}$$

where $i,j=1,2,...,N$, x_i denotes the i-th particle position on a circle of radius L, β is a non-negative real parameter, and V is a regular (*i.e.* finite for every $\{x_i\}$) potential. The parameter β has been found to characterize the *exclusion* statistics of the particles[6, 7]; in particular, (once appropriate boundary conditions are imposed[6]) $\beta=0$ corresponds to bosons and $\beta=1$ corresponds to fermions[2].

The cases $V=0$ ("free Calogero-Sutherland particles") and $V=\sum_{i<j}(x_i-x_j)^2$ ("Calogero-Sutherland particles with an harmonic potential") have been completely solved[10, 11] both for finite L and in the limit $L\rightarrow\infty$.

A very important open problem[5, 6, 8] is the one of finding a formulation of the Calogero-Sutherland models in the formalism of non-relativistic quantum field theory. In the case of anyons[17], particles in 2+1 dimensions that have fractional *exchange* statistics[17], such a formulation is given by a Chern-Simons field theory, and has been very useful[18-20] in the understanding of the statistics.

In this Letter I propose a technique of investigation of the Calogero-Sutherland models which should allow to uncover some of the features of their yet-to-be-found field theoretical formulation. My analysis is indeed motivated by an analogy with the case of the anyon models. In that context it has been recently realized[20, 21] that the ultraviolet structure of the perturbative expansions in the statistical parameter is closely related to the structure of the Chern-Simons field theoretical formulation, which, for example, leads to Feynman diagrams affected by ultraviolet divergences reproducing the ones encountered in the quantum mechanical perturbative framework[20, 21]. I am therefore interested in an analogous perturbative expansion for the Calogero-Sutherland models.
I start by analyzing the ultraviolet problems of such an expansion. For simplicity, I limit the discussion to the case of two Calogero-Sutherland particles with $0\leq\beta\leq1$, an harmonic oscillator potential, and $L\sim\infty$; the relative motion is therefore described by the Hamiltonian

$$H_\beta = -\frac{d^2}{dx^2} + x^2 + \frac{\beta(\beta-1)}{x^2} \ , \tag{2}$$

where x is the relative coordinate, and, since we are dealing with two identical particles on the line, the configuration space is $x\geq0$. The harmonic potential is introduced[9] in order to discretize the spectrum, so that the dependence on β can be examined more easily.
The eigenfunctions of H_β that are regular at the point $x=0$, where the particle positions coincide, are[9] (the L_n^μ are Laguerre polynomials and the N_n^β are normalization constants)

$$|\Psi_{n,\beta}>= N_n^\beta \ x^\beta \ e^{-\frac{x^2}{2}} \ L_n^{\beta-1/2}(x^2) \ , \tag{3}$$

and have energies

$$E_{n,\beta} = 4n+2+2(\beta-1/2) \ . \tag{4}$$

In analogy with the perturbative approaches used in the study of anyons, one can consider perturbative expansions around zero-th order solutions $|\Psi_{n,\beta_0}>$,E_{n,β_0}, which would allow to

[2]Note that the potential $sin^{-2}(\pi x/L)$, which is singular at the points of coincidence of particle positions and causes the fractionality of the *exclusion* statistics, has vanishing coefficient for $\beta=0,1$.

describe the fractional *exclusion* statistics of particles with any β in terms of the one of particles with $\beta=\beta_0$. Important building blocks of such a perturbative expansion are the matrix elements

$$< \Psi_{n,\beta_0}|\frac{1}{x^2}|\Psi_{n,\beta_0} >\sim \int_0^\infty dx \; \frac{e^{-x^2}[L_n^{\beta_0-1/2}(x^2)]^2}{x^{2-2\beta_0}} \; , \tag{5}$$

but these are (ultraviolet) divergent for every $\beta_0 \leq 1/2$. An ultraviolet problem somewhat analogous to this one is encountered in the study of anyons. In that case one is interested in perturbative expansions depending on the statistical parameter ν[17, 23], which also has bosonic limit $\nu = 0$ and fermionic limit $\nu = 1$, and one encounters logarithmic ultraviolet divergences when expanding around the special (bosonic) value $\nu = 0$. This divergent bosonic end perturbation theory of anyons can be handled[23, 24] by using the formalism of renormalization for quantum mechanics[25], and a direct relation between the structure of the renormalized perturbative approach and some features of the Chern-Simons field theoretical formulation of anyons has been found[19-21]. The hope that such a program might be completed also for the Calogero-Sutherland models is confronted by the realization that the ultraviolet problems illustrated by Eq.(5) are much worse than the ones of the anyon case. Rather than being specific of a certain choice of the center of the expansion β_0, these ultraviolet problems are encountered for any of a continuous of choices of β_0, and in general the divergences are worse-than-logarithmic. However, from Eq.(5) one can see that the expansion around $\beta_0=1/2$ is only affected by logarithmic divergences, and therefore this type of expansion is the best candidate for a generalization of the results obtained for anyons with the bosonic end perturbation theory.

Motivated by this observation, I now consider more carefully the possibility of studying Calogero-Sutherland particles with any β as perturbations of "Calogero-Sutherland semions", *i.e.* Calogero-Sutherland particles with $\beta=1/2$. Following the usual path of renormalization theory in quantum mechanics (see, for example, Refs.[20,23-25]), I add (only for the perturbation theory) the counterterm[3] $(\beta-1/2)\sum_{i<j}\delta(x_i-x_j)/(x_i-x_j)$ to the original Calogero-Sutherland Hamiltonian. A very general verification of the validity of this procedure will be given in detail elsewhere[26], but here I intend to briefly describe (to second order in the eigenenergies and first order in the eigenfunctions) how the exact two-body solutions (3) and (4) are correctly reproduced in this way. Let me start by setting up the renormalized quasisemionic description of the H_β eigenproblem. The zero-th order Hamiltonian, wave functions, and energies are obviously the ones for semions, *i.e.* $H_{1/2}$, $|\Psi_{n,1/2}>$, and $E_{n,1/2}$ [see Eqs.(2), (3), and (4)]. The renormalized perturbative Hamiltonian is

$$H_{1/2}^{Rpert} \equiv H_\beta - H_{1/2} + (\beta-1/2)\frac{\delta(x)}{x} = (\beta-1/2)\frac{\delta(x)}{x} + \frac{(\beta-1/2)^2}{x^2} \; . \tag{6}$$

Moreover, the regularization/renormalization procedure obviously requires the introduction of a ultraviolet cut-off Λ, which is ultimately removed by taking the limit $\Lambda \to \infty$. In quantum mechanics such a cut-off is introduced[20,23-25] in the integrals that characterize the matrix elements of the ultraviolet-divergent terms of the perturbative Hamiltonian; for the present case I define

$$\int_0^\infty dx \; \frac{1}{x^2} \; f(x) \equiv \int_{1/\Lambda}^\infty dx \; \frac{1}{x^2} \; f(x) \; , \tag{7}$$

[3] As I shall show in a longer paper[26], the form $\delta(x)/x$ of the counterterm can be derived using symmetries and the appropriate power-counting rules for renormalization in quantum mechanics, and the value of the overall coefficient $(\beta-1/2)$ can be obtained by looking for a fixed point of the renormalization group flow.

$$\int_0^\infty dx \, \frac{\delta(x)}{x} \, f(x) \equiv \Lambda f(1/\Lambda) \, . \tag{8}$$

Since $H_\beta - H_{1/2}$ is quadratic in $(\beta - 1/2)$, for the first order energies one easily finds

$$E^{(1)}_{n,\beta} = < \Psi_{n,1/2}| \, \frac{(\beta - 1/2)}{|x|} \delta(x) \, |\Psi_{n,1/2} >= 2(\beta - 1/2) \, , \tag{9}$$

and from Eq.(4) one sees that $E^{(1)}_{n,\beta} = E_{n,\beta} - E_{n,1/2}$, *i.e.* the first order result (9) already reproduces the exact result, as expected since the latter is linear in $(\beta - 1/2)$.

The first order eigenfunctions are given by

$$\begin{aligned} |\Psi^{(1)}_{n,\beta} > &= \sum_{m(\neq n)} \frac{< \Psi_{m,1/2}| \, \frac{(\beta-1/2)}{|x|} \delta(x) \, |\Psi_{n,1/2} >}{E_{n,1/2} - E_{m,1/2}} |\Psi_{m,1/2} > \\ &= -\frac{(\beta - 1/2)}{2\sqrt{\pi}} \sum_{m(\neq n)} \frac{L^0_m(x^2)}{m - n} x^{1/2} \, e^{-\frac{x^2}{2}} \, , \end{aligned} \tag{10}$$

which, as it can be seen using properties of the Laguerre polynomials, is in agreement with the expansion of Eq.(3) to first order in $(\beta - 1/2)$.

Concerning the second order energies, I have verified that

$$< \Psi_{n,1/2}| \, \frac{(\beta - 1/2)^2}{x^2} \, |\Psi_{n,1/2} >= - < \Psi_{n,1/2}| \, \frac{(\beta - 1/2)}{|x|} \delta(x) \, |\Psi^{(1)}_{n,\beta} > + O(\frac{1}{\Lambda}) \, , \tag{11}$$

when the matrix elements are regularized using the prescriptions (7) and (8). From Eq.(11) one immediately sees that, in agreement with Eq.(4), $E^{(2)}_{n,\beta} = 0$ in the $\Lambda \to \infty$ limit.

This completes the announced two-body test of my quasisemionic description of Calogero-Sutherland particles. Actually, it should be clear that this test illustrates the general mechanism that leads to renormalized (finite) results; indeed, in the study of the many-body problem one simply encounters many identical copies of the same divergence (which originate from many copies of the same two-body $\frac{1}{x^2}$–type interaction), and they obviously require a corresponding number of copies of the same two-body $\frac{\delta(x)}{x}$–type counterterm. This procedure of generalization to the N-body problem has been worked out in detail in the anyon case[23], and will be discussed for Calogero-Sutherland models in [26].

The quasisemionic description that I have proposed and tested in this Letter should be useful in the understanding of 1+1-dimensional fractional *exclusion* statistics, like the corresponding perturbative approach to the study of anyons has been useful in the understanding of (2+1-dimensional) fractional *exchange* statistics. Indeed, both here and in the anyon case the original divergences are due to the singular potentials which cause the fractionality of the statistics, and therefore the regularization/renormalization procedure encodes information on the nature of these statistical interactions. In particular, the knowledge of the structure of the new renormalized perturbative approach should be of help in the ongoing search[5, 6, 8] of a field theoretical formulation of the Calogero-Sutherland models, which, in particular, should have (like Chern-Simons non-relativistic field thery, but unlike the usual non-relativistic field theory scenario) a non-trivial renormalization-requiring ultraviolet structure related to the one I encountered in the present quantum mechanical treatment.

Additional input for the search of a field theoretical formulation could come from devising a quasibosonic perturbative approach to the study of Calogero-Sutherland models, which

would be the ideal starting point for a bosonized field theoretical formulation. Research in this direction is certainly encouraged by my results for the quasisemionic perturbative approach, but, as I showed, the structure of the divergences in the bosonic limit is substantially different from the one of the divergences I regularized/renormalized here.

The special role played by Calogero-Sutherland semions in my analysis should have deeper physical roots (probably related to the special properties of particles with $\beta = 1/2$ pointed out in Refs.[15, 29]) than the rather formal ones I noticed here. In particular, the fact that the renormalizability of the semionic perturbation theory that I considered arises in complete analogy with the renormalizability of the bosonic perturbation theory used for anyons, might suggest that semions play a special role in (1+1-dimensional) exclusion statistics, just like bosons have a special role[4] in (2+1-dimensional) exchange statistics.

From the point of view of mathematical physics it is noteworthy that one more application of renormalization in quantum mechanics has been here found. There are not many such applications and this one should be particularly easy to examine because the problem is 1+1-dimensional and all exact solutions are known. In particular, certain comparisons between the exact solutions and the renormalization-requiring perturbative results might lead to insight in the physics behind the general regularization/renormalization procedure; for example, since the exact solutions (3)-(4) are well-defined at every scale, my analysis is consistent with the idea[20, 24] that the necessity of a cut-off is simply an artifact of the perturbative methods used, and not a relict of some unknown ultraviolet physics.

Finally, I want to emphasize that I chose to consider only the regular Calogero-Sutherland eigenfunctions because they have a clearer physical interpretation[9] and allow a scale-invariant[5] analysis[26], but, based on the experience with anyons[20, 27, 28], I expect that additional insight into the nature of 1+1-dimensional fractional *exclusion* statistics might be gained by looking at the renormalized perturbative expansion of the Calogero-Sutherland eigenfunctions that are singular at the points of coincidence of particle positions.

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[4]The simplest description of particles with fractional exchange statistics is in terms of bosonic fields[19], and only starting from bosons one can obtain, via renormalization group equations (see, for example, Refs.[20, 21]), 2+1-dimensional particles with any given statistics and any given value of the parameter[20, 27] characterizing the consistent (*i.e.* leading to a self-adjoint Hamiltonian) choices of the domain of the Hamiltonian.

[5]As I shall show in a longer paper[26], the scale invariance of the boundary conditions satisfied by the regular wave functions at $x=0$ is related to the fact that here it was appropriate to work at a fixed point of the renormalization group flow and therefore I obtained results which do not involve a renormalization scale.

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